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A. YU. ISHLINSKII'S RESEARCHES IN THE FIELD OF ROLLING FRICTION AND THEIR DEVELOPMENT†

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The results of the research by A. Yu. Ishlinskii and others on the theory of rolling contact of deformable bodies, taking into account their imperfect elasticity and partial slip in the area contact region, are presented. © 2003 Elsevier Ltd. All rights reserved.

A. Yu. Ishlinskii began his scientific career with research on the theory of rolling friction, the results of which were described in his candidate's dissertation (in 1938) [1] and in his first publications [2–4]. He subsequently returned on an analysis of this problem more than once [5, 6]. He was in the first ranks of researchers who analysed the effect of the two basic sources of resistance to rolling of bodies, i.e. the relative slip of the surfaces in the contact region and the hysteresis losses associated with the imperfect elasticity of the materials.

Rolling friction was studied for the first time by Coulomb [7] who, on the basis of experimental investigations, proposed that the dependence of the force of rolling friction T on the radius of the roller R and the load on it P should be calculated using the formula $T = kP/R$, where k is a coefficient which is usually known as the "friction shoulder". This formula was subsequently subjected on many occasions to experimental verification for rollers of different diameters and made of different materials. Under different conditions of friction, a different dependence of the friction force on the mechanical and geometrical characteristics of the interacting bodies was found. This is due to the fact that the factors which give rise to resistance to rolling can be different depending on the properties of the materials and the conditions of the interaction.

The classical work of Reynolds [8] is concerned with a detailed study of rolling friction. The results of research into the rolling of rubber and steel rollers over the flat surfaces of various materials (glass, boxwood, rubber, copper, etc.) were presented in this paper. He established that the actual distance covered by a steel roller after a single revolution in rolling over soft rubber is less than the so-called geometric distance, which is equal to the length of the deconvolution of its surface. Reynolds explained the cause of rolling friction in the cases he studied as being due to the relative slip of points on the surfaces of the interacting materials in separate segments of the contact region, as a consequence of the deformation of the bodies. Petrov [9] also suggested the same cause of the resistance to the motion of wheels on railway track. Partial slip is therefore one of the basic causes of resistance to the rolling of bodies.

Since absolutely elastic bodies do not exist, hysteresis losses in bodies due to their deformation are also a source of resistance to rolling. This mechanism of rolling friction was investigated experimentally by Tabor [10]. In the case of inelastic materials, the rolling friction depends very much on the rate of rolling.

Both of these sources of resistance to rolling, as well as the molecular interaction of the contacting surfaces [11], play a major role in the formation of the force of rolling friction, and the specific value of each of changes, depending on the physicochemical properties of the materials and the external conditions.

1. FORMULATION OF THE PROBLEM OF THE STEADY ROLLING OF DEFORMABLE BODIES, TAKING ACCOUNT OF PARTIAL SLIP IN THE CONTACT REGION

We shall consider the rolling, at an angular velocity ω and linear velocity V directed along the x axis, of a body along the surface of another body (Fig. 1). As a consequence of the deformation of the

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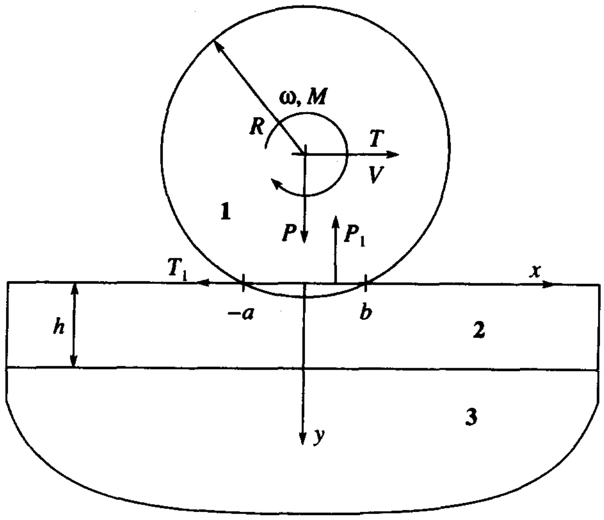

Fig. 1

interacting bodies, there are tangential displacements of a point on the surfaces ($y = 0$) $u_{xi}(x, z, t)$ along the x axis and $u_{zi}(x, z, t)$ along the z axis (which is directed perpendicular to the plane of the sketch) and, also, displacements along the normal to the surface $u_{yi}(x, z, t)$.

The linear velocity of displacement of a particle, located on the surface of the contacting bodies ($i = 1, 2$), which components υ_{xi} along the x axis and υ_{zi} along the z axis in the system of coordinates $Oxyz$ associated with the moving body, is given by the following expressions

$$\upsilon_{xi} = V + \delta V_{xi} + V\frac{\partial u_{xi}}{\partial x} + \frac{\partial u_{xi}}{\partial t}, \quad \upsilon_{zi} = \delta V_{zi} + V\frac{\partial u_{zi}}{\partial x} + \frac{\partial u_{zi}}{\partial t}$$

where δV_{xi} and δV_{zi} are the projections of the slip velocities of the contacting bodies on to the x and z axes.

The velocities of relative slip $s_x(x, z, t)$ in the direction of the x axis and $s_z(x, z, t)$ in the direction of the z axis at a point (x, z) of the contact region are determined by the difference in the velocities of the particles of the interacting bodies at this point, that is,

$$s_x(x, z, t) = \upsilon_{x1} - \upsilon_{x2}, \quad s_z(x, z, t) = \upsilon_{z1} - \upsilon_{z2}$$

In the case of steady rolling, that is, of uniform motion under constant forces, the elastic displacements in the system of coordinates $Oxyz$ are independent of time. In this case, $\partial u_{xi}/\partial t = \partial u_{zi}/\partial t = 0$ and the velocities of the relative slip $s_x(x, z, t)$ and $s_z(x, z, t)$, as well as the components of the displacements and stresses, are solely functions of the x and z coordinates.

In the case of rolling, the whole contact region Ω is divided into two subregions: a subregion Ω_a in which the particles located on the surfaces of the interacting bodies stick and Ω_s where they slip. The boundary conditions in the contact region are written in the form:

in the stick subregion, $(x, z) \in \Omega_a$, there is no slip and the shear stresses $\tau(x, z)$ do not exceed a limiting value, that is

$$s_x = s_z = 0, \quad |\tau(x, z)| \le \mu p(x, z) \tag{1.1}$$

($p(x, z)$ is the contact pressure and μ is the friction coefficient)

in the slip subregion, $(x, z) \in \Omega_s$, the Coulomb–Amonton law

$$|\tau(x, z)| = \mu p(x, z) \tag{1.2}$$

applies and the direction of the shear stresses $\tau(x, z)$ is opposite to the slip direction, that is

$$\frac{\tau(x, z)}{|\tau(x, z)|} = -\frac{s(x, z)}{|s(x, z)|} \tag{1.3}$$

Note that, in the case of complete sliding, the Coulomb–Amonton law (1.2) holds over the whole of the contact region Ω.

The contact condition for the interacting bodies leads to the following relation, which is imposed on the displacement u_{yi} of a point of the surface in a direction normal to it

$$u_{y1} + u_{y2} = D - f_1(x, z) + f_2(x, z), \quad (x, z) \in \Omega \tag{1.4}$$

where $f_1(x, z)$ and $f_2(x, z)$ are the shape equations of the interacting bodies and D is the approach of the bodies in the direction of the y axis as a result of their deformation.

In order to determine the stress distribution and the rolling resistance it is necessary to solve a contact problem with the boundary conditions presented above. The major difficulty in solving it arises in determining the positions both of the boundaries of the stick and slip zones (there can be several stick and slip zones which the contact region).

2. THE ROLLING OF ELASTIC BODIES

The stressed state of elastic bodies and the resistance to rolling depend on the difference in the curvatures of the bodies in the contact region and the ratio of the moduli of elasticity. Two geometrically identical elastic bodies with the same elastic characteristics do not experience resistance to rolling under the action of just a normal force, and, at the same time, partial slip does not occur in the contact region.

The two-dimensional contact problem of the rolling of an elastic cylinder of radius R along a base made of the same material under the action of a moment M and a tangential force T, was investigated in [12–16].

The forces and moments acting on a roller (a cylinder), separated into active and reactive forces, are shown in Fig. 1. The moment M is in the direction of rotation if the roller is a drive roller and in the reverse direction in the case of a slave or braking roller. The contact pressure $p(x)$ and the shear stress $\tau(x)$ constitute reactive forces. All the forces are assumed to be constant along the generatrix of the cylinder. The equations

$$T = \int_{-a}^{b} \tau(x)dx, \quad P = \int_{-a}^{b} p(x)dx, \quad TR = \int_{-a}^{b} xp(x)dx + M \tag{2.1}$$

hold in the case of the uniform motion of a slave roller, where P is the vertical force acting on the roller, and $-a$ and b are the boundaries of the contact region.

If the materials of the interacting bodies are the same, the shear stresses do not have any effect on the distribution of the contact pressure and the sizes of the contact regions, which are determined using Herts theory. Partial slip in the contact region occurs due to the difference in the curvatures of the interacting bodies. It was shown [12–14] that, in the case of the same elastic materials of contacting bodies, only two zones can be formed in the contact region: the stick zone located in front, on the side where the roller approaches the base, and then the slip zone. The following shear stress distribution $\tau(x)$ in the contact region $(-a, a)$ was obtained in [12]

$$\tau(x) = \frac{\mu}{\pi KR}\begin{cases} \sqrt{a^2 - x^2}, & -a \le x \le c \\ [\sqrt{a^2 - x^2} - \sqrt{(a-c)(x-c)}], & c \le x \le a \end{cases}; \quad K = \frac{2(1-\nu^2)}{\pi E} \tag{2.2}$$

which c is the point where the zone stick changes into the slip zone, E is Young's modulus and ν is Poisson's ratio of the interacting bodies. The relations

$$2a = \sqrt{8KRP}, \quad \frac{a-c}{2a} = 1 - \frac{\delta R}{\mu a}; \quad \delta = \frac{\omega R - V}{V} \tag{2.3}$$

have been obtained for the half-width of the contact region and the size of the stick zone where ω is the angular velocity of rotation of the cylinder and δ is the creep ratio.

The shear stress distribution, calculated using formulae (2.2) and (2.3) for $T/(\mu P) = 0.5$ is represented by curve 1 in Fig. 2. The shear stress distribution in the case of full slip $\tau(x) = \mu p(x)$ is shown by the dashed curve 1.

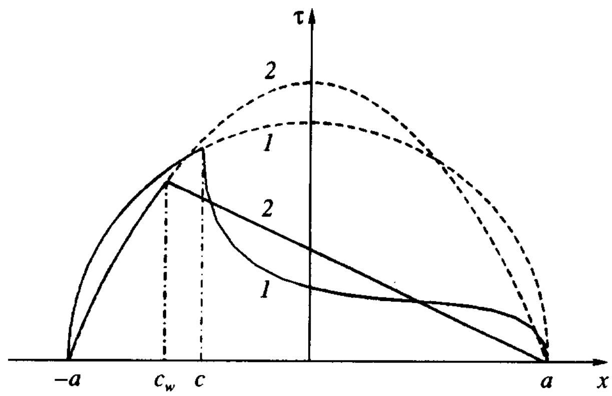

Fig. 2

The shear force T is connected with the relative slip δ by the relation

$$\frac{T}{\mu P} = \frac{\delta R}{\mu a}\left(2 - \frac{\delta R}{\mu a}\right) \tag{2.4}$$

Experimental data [16] confirm the location of the stick and slip zones in the contact region obtained in [12–14]. However, in an investigation of the problem of the interaction of two rotating discs of different radii made of the same elastic material, it was shown [17] that, in the case of certain values of the external forces, a rolling scheme with three zones in the contact region (a stick zone surrounded by two slip zones) is feasible.

In the case of rolling of two elastic bodies made of different materials, additional slip occurs due to the difference in the shear deformations on the interacting surfaces as a consequence of the difference in their elasticity constants.

Ishlinskii's research [1, 3] was concerned with investigating the rolling contact of a rigid cylinder with an elastic base. He used a simplified model of the base (an extension of the Winkler model to the compliance accompanying shear), in which the normal u_y and tangential u_x displacements of point on the surface are connected with the pressure p and the shear stresses τ acting in the contact region by the relations

$$u_y = (h/K_n)p, \quad u_x = (h/K_\tau)\tau \tag{2.5}$$

Substituting these expressions into boundary condition (1.1)–(1.4) one can determine the normal and shear stress distributions in the contact region and the arrangement of the stick and slip zones in this region. In particular, the pressure within the contacting region in this case has a parabolic distribution, and the shear stress distribution in the stick zone is linear, which follows from the solution of the ordinary differential equation in the stick zone with conditions of continuity of the stresses at the points where the stick zone changes into the slip zone or this zone merges into the boundary of the contact region. A graph of the shear stress distribution τ in the contact region when there is a single stick zone (c_w, a) and a single slip zone $(-a, c_w)$ is shown in Fig. 2 (curve 2).

A full analysis of the positions of the stick and slip zones in the contact region and the conditions under which a scheme with two zones (a stick zone which is in front, at the leading edge of the contact region, changes into a slip zone) and three zones (a stick zone is located between two slip zones) in the contact region is feasible, has been carried out [4]. Relations are also given which enable one to calculate the resistance to rolling for large and small values of the moment M as well as an approximate formula for the maximum value of the force of rolling friction.

The simplified model of an elastic base used by Ishlinskii to analyse the contact characteristics in the problem of the rolling of a cylinder (the two-dimensional formulation) was subsequently used to solve three-dimensional problems of the rolling of elastic bodies and, also, to investigate unsteady rolling problems and the transient to steady rolling [18, 19]. An analysis of the possibility of using simplified model (2.5) to investigate contact characteristics in the rolling of elastic bodies, carried out in [19] by comparing the results of the solution of problems for an unsimplified model of an elastic base (analytical and numerical) with the simplified model, showed that the simplified model enables one to calculate the dimensions and locations of the stick and slip zones in the contact region with sufficient accuracy (the error is no greater than 15%), as well as the magnitude of the relative slip accompanying rolling.

Whereas, in the case of identical elastic properties of contacting bodies, the problem of the rolling of a cylinder along a base can reduced to Riemann–Hilbert problem for one analytic function (see below, Section 3), the solution of which has the form of (2.2), in the case of dissimilar elastic properties, it is necessary to solve the related problem of finding two analytical functions. The problem of an elastic cylinder rolling on an elastic base, when there are no constraints whatsoever imposed on the elastic properties of the cylinder and the base, has been considered in [20] under the assumption that the contact region consists of two zones, that is, of slip and stick zones, and, in the stick zone, it was assumed, as in [15], that the tangential displacements of the interacting surfaces are equal. The solution was reduced to investigating the problem of liner matching for two analytical functions, which was solved by the setting up a Gauss differential equation with three singular points. The final expression for the stresses is rather complex, which made it difficult to carry out any further analysis of the results. Hence, the analysis of the friction forces in the problem of a rigid cylinder rolling on an elastic base and the arrangement of the zones where stick and slip occurs in the contact region, which was carried out for the first time by Ishlinskii [4] is up to now unique in the investigation of the rolling of the bodies made of materials with quite different elastic moduli.

The three-dimensional problem of the rolling of elastic bodies has been thoroughly investigated in the Kalker's monograph [19], which also contains a historical review of publications in the field of the rolling friction of elastic bodies. A variational approach is used to solve three-dimensional contact problems [21]. This approach consists of finding the minimum, in the space of permissible functions of the shear contact stresses, of a functional of the form

$$I = \int_{\Omega} (|\dot{s}|\tau + \mu p \dot{s})^2 dS \tag{2.6}$$

The equivalence of the variational formulation (2.6) to the problem of the contact of rolling bodies made of identical materials with boundary conditions (1.1)–(1.4) has been proved [22, 23]. When the variational approach is used, the unknown boundaries of the stick and slip zones are constructed, after solving the variational problem, using the velocity field which has been found. In a numerical implementation, the variational problem is approximated by a discrete problem of non-linear programming [23].

The results of the numerical solution showed that the shape of the contact region accompanying the rolling of a sphere on a plane is almost circular. An analysis of the solution [19] in the case of contacting bodies made of identical materials established that the shear stress distribution in a line passing through the centre of the contact region and collinear with the direction of the action of the traction force T is close to the distribution shown in Fig. 2. The results obtained served as a basis for the use of approximate methods, the basis of which is the superpositioning of the solutions of the two-dimensional rolling contact problem, in solving three-dimensional rolling problems. Thus, in the band theory [24], the contact region is divided into thin bands parallel to the direction of rolling. The solution of the problem in a two-dimensional formulation is used for each such band, neglecting the interactions between them. A review of the different methods of solving three-dimensional rolling problems for elastic bodies made of identical materials, on the assumption that the size of the contact region is much smaller than the radii of curvature of the interacting bodies, has been given in [23].

3. THE ROLLING OF VISCOELASTIC BODIES

Loading and unloading of the interacting bodies occurs as they roll, which, by virtue of the rheological properties of the materials, leads to hysteresis losses.

In 1938, the problem of a rigid cylinder (a roller) rolling on a viscoelastic base under steady conditions was considered by Ishlinskii for the first time. The solution of this problem enables one to calculate the moment of rolling friction and to investigate its dependence on the rolling velocity, the load, as well as the mechanical and geometrical characteristics of the interacting bodies [2]. Two one-dimensional models of the material were used in the approximate solution of the problem. In these models, the pressure $p(x)$ in the contact region is associated with displacement $u_y(x)$ of the surface along the normal to it by the relations

$$p(x) = K_n u_y(x) + \kappa \partial u_y(x)/\partial t \tag{3.1}$$

$$\partial u_y(x)/\partial t = K_1 p(x) + k_2 \partial p(x)/\partial t \tag{3.2}$$

where K_n, κ, K_1 and K_2 are constants of the material.

In solving the problem, it was assumed that there are no shear stresses in the contact region and that the rolling resistance arises solely due to the pressure asymmetry, which leads to the appearance of a moment of resistance. It was shown that, in the case of the model (3.1) which possesses a bounded creep, the rolling resistance force, at low rolling velocities, is described by the relation

$$T = \frac{\kappa VP}{K_n R} \tag{3.3}$$

and, in the case of high rolling velocities, by

$$T = \frac{4}{3}\left[\frac{P^3}{2\kappa VRl}\right]^{1/2} \tag{3.4}$$

It is curious to note that these two asymptotic formulae, had they been obtained much earlier, could have put an end to the heated discussion [25] which arose between Dupuit and Moren at the end of the nineteenth century in connection with how the rolling resistance depends on the radius of the roller. While Dupuit reckoned that the force T is proportional to $R^{-1/2}$, Moren defended another rule: the force T is proportional to R^{-1}.

In the case of a viscoelastic soil, which obeys the law of deformation (3.2), the dependence of the friction force on the mechanical and geometrical characteristics of the contacting bodies at high rolling velocities V obtained by Ishlinskii has the form

$$T = \frac{K_1}{5V}\left[\frac{18P^5}{K_2 Rl^2}\right]^{1/3} \tag{3.5}$$

One-dimensional (rod) models for describing the imperfect elasticity of the base when cylindrical and spherical bodies roll over it were used later in [26, 27], where the behaviour of a rod under compression was described by various first-order differential equations.

The solution of the problem of a rigid cylinder rolling on a base, which is described by a model of a viscoelastic continuum, has been obtained in [28], also with the assumption that there are no shear forces in the contact region. The simplest linear medium, for which the relaxation function $\gamma(t)$ has the form

$$\gamma(t) = \frac{1}{G_D}(1 + \beta(1 - e^{-t/T}))$$

was adopted as the model of the material of the base, where G_D is the dynamic shear modulus and T is the relaxation time. The contact between two cylinders with different and identical elasticity constants has been considered in [29, 30] for the same viscoelastic materials. In this case, solutions were found which correspond to a spectrum of relaxation times. It was established that the rolling resistance force has a maximum when retardation time of the material is comparable with the contact time. The distribution of the normal stresses accompanying the rolling of a viscoelastic cylinder on a base made of the same material has been found in [31], for which the relation between the stresses and strains was expressed by Volterra integral relations with an exponential kernel.

The problem of a viscoelastic cylinder rolling on a base of the same material was treated under the assumption that the relations between the stresses and strains in a viscoelastic body have the form

$$\varepsilon_x^* = \frac{1-\nu^2}{E}\sigma_x^* - \frac{\nu(1+\nu)}{E}\sigma_y^*, \quad \varepsilon_y^* = \frac{1-\nu^2}{E}\sigma_y^* - \frac{\nu(1+\nu)}{E}\sigma_x^*, \quad \gamma_{xy}^* = \frac{1+\nu}{E}\tau_{xy}^* \tag{3.6}$$

where

$$\varepsilon_{ij}^* = \varepsilon_{ij} - T_\varepsilon V\frac{\partial \varepsilon_{ij}}{\partial x}, \quad \sigma_{ij}^* = \sigma_{ij} - T_\sigma V\frac{\partial \sigma_{ij}}{\partial x}, \quad i = x, y \tag{3.7}$$

Here T_ε and T_σ characterize the viscous properties of the material, ν is Poisson's ratio and E is the long-term modulus of elasticity of the material. For the model under consideration, the instantaneous modulus of elasticity is determined by the quantity $H = \alpha E$, where $\alpha = T_\varepsilon/T_\sigma$. Note that, in the case of amorphous polymers, $\alpha = 10^5$–10^7, for polymers with a high degree of crystallinity $\alpha = 10$–10^2, and for ferrous metals, $\alpha = 1.1$–1.5.

It is assumed in the formulation of the problem [32] that the whole contact region $(-a, b)$ is subdivided into a zone where slip occur $(-a, c)$, and stick zone (c, b) located in front in the direction of the incidence of the cylinder. The boundary conditions (1.1)–(1.4) enabled the problem to be reduced to the determination of two functions which are analytical in the lower half-plane

$$w_1(z) = \int_{-a}^{b} \sigma_y^*(t, 0)\frac{dt}{t-z} = U_1(x, y) - iV_1(x, y) \tag{3.8}$$

$$w_2(z) = \int_{-a}^{b} \tau_{xy}^*(t, 0)\frac{dt}{t-z} = U_2(x, y) - iV_2(x, y) \tag{3.9}$$

the real and imaginary parts of which satisfy the following conditions on the boundary ($y = 0$)

$$\begin{aligned} &V_1 = V_2 = 0, \quad x \notin (-a, b) \\ &U_1 = -\frac{x - T_\varepsilon V}{2KR}, \quad V_2 + \mu V_1 = 0, \quad x \in (-a, c) \\ &U_1 = -\frac{x - T_\varepsilon V}{2KR}, \quad U_2 = \frac{\delta}{2K}, \quad x \in (c, b) \end{aligned} \tag{3.10}$$

The quantities K and δ are defined by the last relations of (2.2) and (2.3). The boundary conditions (3.10) initially enable one to solve the problem of determining the function $w_1(z)$ (3.8) and, subsequently, enable one to determine the function $w_2(z)$ (3.9) using the function $V_1(x, 0)$ which has been found. The functions ε_{ij}^* and σ_{ij}^* on the real axis are expressed in terms of the real imaginary parts of the functions $w_1(z)$ and $w_2(z)$. The true stresses and displacements which act on the boundary of the half-plane are then found from the solutions of differential equations (3.7). Analytical expressions for the normal and shear contact stresses, as well as the equations for determining the dimensions of the stick and slip zones, have been obtained in [32, 33].

An investigation of the problem in [33] enabled one to identify the dimensionless parameter $\zeta_0 = l_0/(2T_\varepsilon V)$, which has a substantial effect on the pressure distribution, the size and displacement of the contact region, etc. Here, $l_0 = \sqrt{8KRP}$ is the width of the contact region for elastic bodies characterized by the elasticity constants E and ν. Diagrams of the pressure distribution $p(x) = -\sigma_y(x)$ for $\alpha = 5$ and different values of the parameter ζ_0 are shown in Fig. 3. At low rolling velocities when the time of passage of the contact region is greater than the retardation time of the material ($\zeta_0 \gg 1$), the pressure distribution and the size of the contact region approach the values corresponding to the case of an elastic material with a long-term modulus of elasticity. At high velocities ($\zeta_0 \ll 1$), the pressure distribution and the size of the contact region again approach the values corresponding to the case of an elastic material, but with an instantaneous modulus of elasticity. When $\zeta_0 \sim 1$, the greatest displacement of the contact region relative to the axis of symmetry of the cylinder is observed and, at the same time, the asymmetry of the diagram of the pressure on it also increases.

As a result of the pressure asymmetry, a moment of rolling resistance M arises. The characteristic dependence of the coefficient of rolling friction $\mu_r = M/(PR)$ on the parameter ζ_0 for the case of free rolling ($T = 0$) is shown in Fig. 4 for different value of α. It follows from an analysis of the results that the relaxation and retardation of the materials of the interacting bodies manifest themselves during rolling when the time of passage of the contact region is comparable with the relaxation time ($\zeta_0 \sim 1$), which is in agreement with the conclusions drawn in [30, 31] and, also, with the dependence of the force of rolling friction on the velocity of motion of the roller obtained by Ishlinskii [2].

Graphs of the dependence of the length of the stick zone on the parameter ζ_0, obtained in [32, 33] when solving the problem of a viscoelastic cylinder of radius R rolling on a base made of the same material when there is partial slip in the contact region are shown in Fig. 5 for different values of the parameter $T^* = T/(\mu P)$. The length of the stick zone increases as T^* decreases and, also, when the parameter ζ_0 increases.

The dependences of parameter T^* on the relative slip δ (creep curves) are shown in Fig. 6 for different value of the parameter ζ_0, $T \geqslant 0$. The curves were constructed for $\alpha = 10$ and different values of the parameter ζ_0. It can be concluded from the results of the calculations that, for a fixed value of $T/(\mu P)$, the magnitude of the relative slip decreases as the parameter ζ_0 decreases (as the velocity V increases).

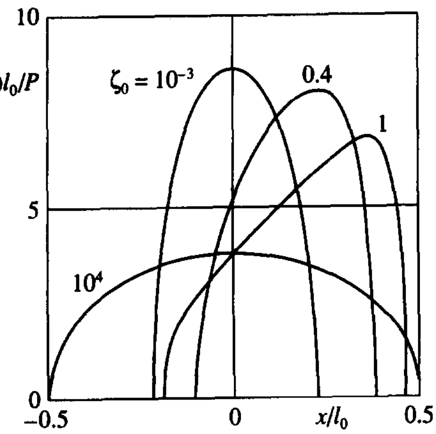

Fig. 3

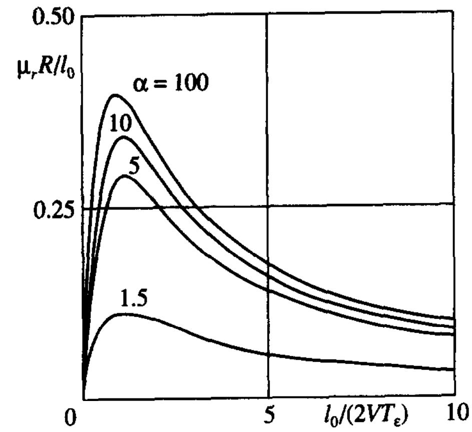

Fig. 4

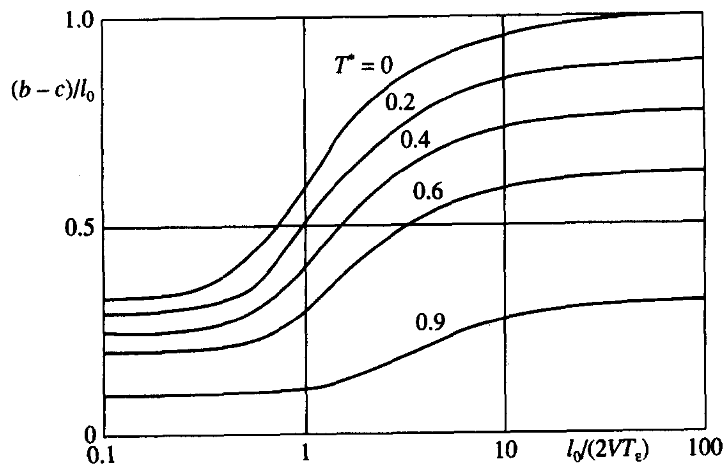

Fig. 5

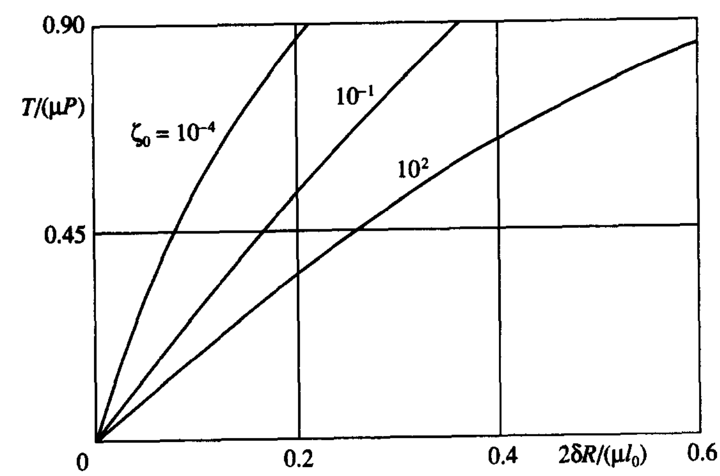

Fig. 6

Moreover, the curve constructed for $\zeta_0 = 10^2$ practically coincides with the graph of the function (2.4), which corresponds to an elastic cylinder, with a modulus of elasticity E and Poisson's ratio ν, rolling on a base made of the same material.

The solution of the problem being considered [22] enabled the combined effect of the two basic sources of rolling friction, that is, the relaxation and retardation of materials, as well as the partial slip of the interacting surfaces in the contact region (and, correspondingly, the coefficient of sliding friction) on the contact characteristics and the coefficient of rolling friction to be considered for the special case when the interacting bodies are made of identical materials.

Other solutions of contact problems, in different formulations, of the rolling of bodies made of viscoelastic materials are also known [18, 34].

4. THE ROLE OF THIN SURFACE LAYERS IN ROLLING FRICTION

The properties of the surface and of surface layers which are substantially different from the properties of the bulk material have a significant effect on the friction characteristics. Under practical conditions (on railway tracks, for example) the observed slip coefficients are lower than those determined theoretically, which is explained, in particular, by the presence on the interacting surfaces of thin films of oils or other types of contaminants [35]. The solutions of contact problems for layered media are used when analysing surfaces covered by thin solid layers or films. In this case, the rheological properties of the surface layers are taken into account when formulating the contact problems by modelling the surface layer by a viscoelastic medium. A problem in two-dimensional formulation on the motion of a load along the boundary of a viscoelastic strip bonded to a viscoelastic half-plane has been treated in [36] using the Fourier transform method, and the strains and shear stresses in the layer and the base were investigated. The contact interaction in the rolling of two cylinders coated with viscoelastic layers has been studied theoretically and experimentally in [37, 38]. In these papers, numerical methods were developed to determine the stresses in contact problems for layered elastic and viscoelastic bodies. Note that the solution of the problem of a rigid cylinder rolling along a viscoelastic base obtained by Ishlinskii [2] enables one to estimate the effect of the rheological properties of the surface layer on the force of rolling resistance, if it is assumed that the modulus of elasticity of the base is much greater than the modulus of elasticity of the layer (that is, assuming that the base is absolutely rigid).

A contact problem in a two-dimensional formulation for an elastic cylinder and a base, consisting of a viscoelastic layer 2 of thickness h bonded to an elastic half-plane 3 (Fig. 1) has been considered in [39, 40]. The cylinder rolls at a constant linear velocity V and angular velocity ω. The contacting surface of the cylinder is described by the function $f(x) = -x^2/(2R)$ (R is the radius of the cylinder).

In the case of the steady motion of the cylinder in the system of coordinates (x, y), the boundary conditions in the contact region have the form (1.1)–(1.4). In order to describe the normal and tangential compliance of the layer, assuming that the thickness h of the viscoelastic layer is much less than the width $(a + b)$ of the contact region, the one-dimensional Maxwell model was used, namely

$$\dot{u}_x = \frac{h}{E_\tau}\left(\frac{\tau}{T_\tau} + \dot{\tau}\right), \quad \dot{u}_y = \frac{h}{E_n}\left(\frac{p}{T_n} + \dot{p}\right) \tag{4.1}$$

where u_x and u_y are the displacements of the boundary of the layer along the tangent and normal to the surface ($y = 0$), a dot denotes a time derivate, and $E_n(E_\tau)$ and $T_n(T_\tau)$ are the modulus of elasticity and the relaxation time of the layer in the direction of the y axis (x axis) respectively. This model is an analogue of the rod model of an elastic body proposed by Ishlinskii [3].

In order to determine the contact pressure distribution $p(\xi)$, the length L and the shift ε of the contact region, as well as the maximum penetration $\Delta_{\max}$ of the cylinder into the viscoelastic layer, a Fredholm integral equation of the second kind was obtained

$$\int_{-1}^{1} F(\xi')\left[\ln|\xi - \xi'| + \frac{\alpha_n}{2}\operatorname{sgn}(\xi - \xi') - \frac{1+\xi'}{2}\ln(1+\xi') - \right.$$
$$\left. - \frac{1-\xi'}{2}\ln(1-\xi') + \frac{\alpha_n \xi'}{2}\right]d\xi' - \frac{\beta_n}{2}F(\xi) = \xi L, \quad -1 \le \xi \le 1 \tag{4.2}$$

as well as the relations

$$\tilde{P} = \frac{2P}{\pi RE^*} = -L\int_{-1}^{1} F(\xi)d\xi, \quad F(\xi) = \tilde{p}'(\xi)$$

$$\varepsilon = \frac{1}{2L}\int_{-1}^{1} F(\xi)[(1+\xi)\ln(1+\xi) + (1-\xi)\ln(1-\xi) - \alpha_n\xi]d\xi \tag{4.3}$$

$$\Delta_{\max} = \max_{-a<x<b} \frac{u_3(x)}{R} = L \max_{-1<x<1} \Phi(\xi)$$

where

$$\alpha_n = \frac{h\pi E^*}{2VE_nT_n}, \quad \beta_n = \frac{h\pi E^*}{2RE_n}, \quad L = \frac{a+b}{2R}, \quad \varepsilon = \frac{b-a}{a+b}$$

$$E^* = \left(\frac{1-\nu_1^2}{E_1} + \frac{1-\nu_3^2}{E_3}\right)^{-1}, \quad \tilde{p}(\xi) = \frac{2}{\pi E^*}p\left(\frac{b-a}{2} + \frac{a+b}{2}\xi\right) \tag{4.4}$$

Here E_i and ν_i are the moduli of elasticity and Poisson's ratios of the materials of the cylinder ($i = 1$) and of the base ($i = 3$). The solution of Eq. (4.2) is presented and an analysis of the results is given in [39–41].

Note that, if the elasticity of the cylinder and the base is neglected and the pressure in the contact region is determined from the solution of Eq. (4.1) with boundary conditions (1.4), we obtain the following expression for the dimensionless contact pressure

$$\tilde{p}(\xi) = \frac{L}{\alpha_n}\left[\mathrm{cth}\frac{1}{\tilde{\varsigma}} + \xi - \frac{\exp(\xi/\tilde{\varsigma})}{\mathrm{sh}(1/\tilde{\varsigma})}\right], \quad \tilde{\varsigma} = \frac{\beta_n}{\alpha_n L} = \frac{2T_nV}{a+b} \tag{4.5}$$

where $\tilde{\varsigma}$ is the Deborah number, which is the ratio of the relaxation time T_n of the material of the layer to the time required for an element to traverse half the width of the contact region, that is, $(a + b)/2$ (see [18]). Expression (4.5) determines the distribution of the contact pressure in the case when the compliance of the layer in the normal direction is much greater than the compliances of the base and the cylinder that is, $E_n/E^* \ll 1$.

Graphs of the function of the contact pressure $p(\xi)/p_0$, where $p_0 = E^*L/2$ is the Hertz maximum contact pressure, constructed for $\beta_n = 0.1$ and different values of the parameter α_n, are shown in Fig. 7. The solid curves correspond to the general case of the contact interaction of elastic bodies when there is a viscoelastic layer between them, and the dashed curves were constructed using formula (4.5) in the case when the elastic properties of the indentor and the base are neglected. Calculation were carried out for a constant width of the contact region $L = 0.1$, and the load acting on the cylinder was varied. The results show that, as the velocity V of the indentor is reduced, that is, as the parameter α_n increases (see (4.4)), the pressure distribution diagram $p(\xi)$ becomes more asymmetrical. In the case of a fixed contact region and specified viscoelastic characteristics of the layer, the contact pressures and their maximum values depend very much on the elastic properties of the indentor and the base when the parameter α_n is small (high velocities V). However, when the velocity is reduced ($\alpha_n = 10$), the difference between the pressure distribution in the two cases becomes negligibly small. A viscoelastic layer therefore has a decisive effect on the distribution of the contact pressure at low velocities of motion.

It was concluded on the basis of the results of calculations that, as the parameter T_nV/R is increased, the half-width of the contact region decreases and tends to a constant value. For small values of the parameter T_nV/R, the length of the contact region becomes substantially greater, particularly when the parameter β_n increases, which depends on the thickness of the layer and the relative elastic characteristics of the layer and the base. As the relaxation time T_n becomes shorter and the velocity V of motion of the indentor is reduced, the displacement ε of the contact region and the maximum penetration $\Delta_{\max}$ of the cylinder into the viscoelastic layer increase, which is due to the manifestation of the rheological properties of the surface layer. When the relaxation time or the velocity V are increased, the shift ε of the contact region becomes negligibly small for all values of the parameter β_n.

The above analysis of the contact pressure distribution and also of the position and dimensions of the contact region, holds both for slip and for rolling of a cylindrical elastic indentor on an elastic base coated with a thin viscoelastic layer.

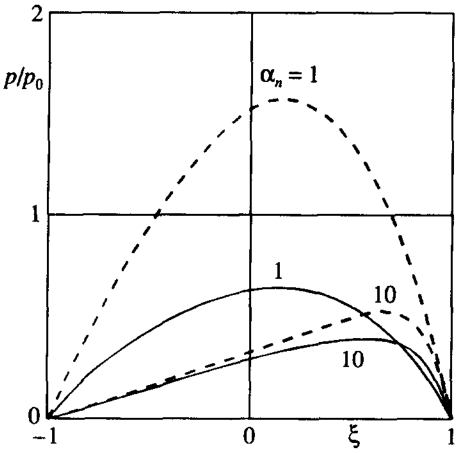

Fig. 7

To determined the shear stress distribution $\tau(x)$ in the stick zone (Ω_a), the following integral equation was obtained [39]

$$-\frac{h}{VE_\tau T_\tau}\tau(x) + \frac{h}{E_\tau}\frac{d\tau(x)}{dx} - \frac{2}{\pi E^*}\int_{-a}^{b}\frac{\tau(x')}{x-x'}dx' = \delta - \frac{2\vartheta}{\pi E^*}p(x), \quad x \in \Omega_a \tag{4.6}$$

where

$$\vartheta = \frac{\pi E^*}{2}\left[\frac{(1-2\nu_1)(1+\nu_1)}{E_1} - \frac{(1-2\nu_3)(1+\nu_3)}{E_3}\right]$$

This equation was reduced [4] to the following Fredholm integral equation of the second kind for determining the function $q(\varsigma) = \tau'(\varsigma)$

$$\varphi(\xi) = 0, \quad \xi \in \Omega_a \tag{4.7}$$

where

$$\varphi(\xi) = -\delta + \vartheta\tilde{p}(\xi) + \frac{\beta_\tau}{L}q(\xi) - \int_{-1}^{1}\left[\ln|\xi-\xi'| + \frac{\alpha_\tau}{2}\text{sgn}(\xi-\xi')\right]q(\xi')d\xi'$$

$$\tilde{\tau}(\xi) = \frac{2}{\pi E^*}\tau\left(\frac{b-a}{2} + \frac{a+b}{2}\xi\right), \quad \alpha_\tau = \frac{h\pi E^*}{2VE_\tau T_\tau}, \quad \beta_\tau = \frac{h\pi E^*}{2RE_\tau} \tag{4.8}$$

Furthermore, the shear stresses satisfy the inequality $|\tilde{\tau}(\xi)| < \mu\tilde{p}(\xi)$ in the stick zone (Ω_a). From Eq. (4.7) and the conditions that, in the slip zones (Ω_s), the shear stresses are opposite to the slip direction, that is

$$\tau(\xi) = \mu\tilde{p}(\xi)\text{sgn}\varphi(\xi), \quad \xi \in \Omega_s \tag{4.9}$$

and that the condition of continuity of the stresses at the points $\xi_i (i = 1.2, \ldots, k)$ of transition from one zone to another, where $(k + 1)$ is the total number of stick and slip zones, holds, an algorithm was constructed for calculating the shear stress in the contact region as well as for calculating the arrangement and sizes of the stick and slip zones [39]. An interactive process was used for the numerical analysis of the relations obtained.

The problem of determining the shear stress is simplified considerably by assuming that the cylinder and the base have the same constants of elasticity ($\vartheta = 0$) and that the modulus of elasticity of the layer is far less than the modulus of elasticity of the cylinder and the base, that is, $E_\tau/E^* \ll 1$. In this case, the problem reduces to investigating ordinary differential equations, and its solution can be written in a simple analytical form [40].

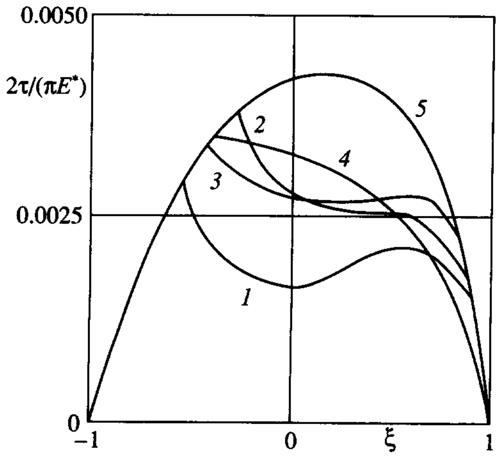

Fig. 8

Analysis of the solution showed that, depending on the magnitude of the relative slip, the coefficient of sliding friction, the mechanical characteristics of the layer and the conditions of interaction (velocity, magnitude of the load and tangential force), the contact region can have two (stick and slip) or three (slip, stick and slip) zones, which distinguish the solution from the case of interaction when there is no viscoelastic layer and only two zones (stick and slip) in the contact region when an elastic cylinder rolls on a base made of the same material ($\vartheta = 0$) [42].

The results of a calculation of the shear stresses in the contact region of a rolling cylinder with a base when there is a surface layer on it, based on an analysis of Eqs (4.7)–(4.9), are shown in Fig. 8. In this analysis, the properties of the viscoelastic layer are described by the parameter $\theta = T_\tau/T_n$, which is a ratio of the relaxation times of the layer in the tangential and normal directions ($\theta = (\beta_\tau\alpha_n)/(\beta_n\alpha_\tau)$) and also by the dimensionless parameters α_n, β_n and β_τ (see (4.4) and (4.8)).

The graphs were constructed for $\beta_n = 0.1$, $\alpha_n = 1$, $\tilde{P} = 0.01$, $\mu = 0.1$, $\beta_\tau = 0.1$ and the following combinations of other parameters: $\tilde{T} = 0.6\mu\tilde{P}$, $\theta = 0.1$, $\vartheta = -0.4$ (curve *1*), $\tilde{T} = 0.8\mu\tilde{P}$, $\theta = 1$, $\vartheta = -0.4$ (curve *2*), $\tilde{T} = 0.8\mu\tilde{P}$, $\theta = 0.1$, $\vartheta = -0.4$ (curve *3*), $\tilde{T} = 0.8\mu\tilde{P}$, $\theta = 0.1$, $\vartheta = 0.4$ (curve *4*) and $\tilde{T} = \mu\tilde{P}$ (curve *5*). The results show that, as the parameter θ increases, the value of the maximum contact shear stresses increase and the size of the stick zone decreases. For the same characteristics of the layer ($\beta_\tau = 0.1$ and $\theta = 0.1$), a change in the elastic characteristics of the cylinder and base from $\vartheta = -0.4$ (curve *3*) to $\vartheta = 0.4$ (curve *4*) involves a transition from three-zone contact to two-zone contact. Moreover, it has established that, as the magnitude of the tangential force $\tilde{T}$ is reduced, the contact changes from complete sliding (curve *5*) to the three-zone case and then to the two-zone case.

The normal and shear contact stress distributions ($|\xi| \leqslant 1$) have been used [43] to determine the stresses σ_x on the surface of the base ($y = h$) for a friction coefficient $\mu = 0.5$ when $\alpha_n = 10$, $\zeta = 0.25$, $\zeta_\tau = 0.1$, $L = 0.1$ and different values of the ratio T/P of the tangential force to the normal force (Fig. 9). As in the case of contact without a viscoelastic layer, the maximum tensile stress, in the case of a non-zero friction coefficient, occurs on the edge of the contact region when $x = -a$($\xi = -1$) and the maximum compressive stress occurs within the contact region. The values of these maxima become greater as the horizontal load increases and, consequently, as the relative slip increases. The fact that there are no corner points on the curves at the places where a stick zone changes into a slip zone, which do occur on the analogous curves in the problem of a cylinder rolling on an elastic base without a surface layer, may explain the effect of a viscous layer. Moreover, when there is a surface layer, the maximum values of the tensile stresses are less than when there is no such layer, that is, the layer reduces the values of the maximum stresses which are capable of initiating the onset of the fracture of the material.

When $\tilde{T} = \mu\tilde{P}$, slip occurs over the whole contact region. The case when $T = 0$ corresponds to pure rolling. Note that, in the case of the model of a viscoelastic layer being considered (a Maxwellian body), the coefficient of rolling friction decreases monotonically as the parameter T_nV/R increases and $\mu_r \to 0$ as $T_nV/R \to +\infty$.

The analysis carried out in [39, 41] shows that the imperfect elasticity of the surface layer has a considerable effect on the contact stress distribution accompanying the sliding and rolling of elastic bodies, particularly for small values of the parameter T_nV/R and, at the same time, the dependence of the resistance to relative displacement of the bodies on the velocity is defined by the rheological

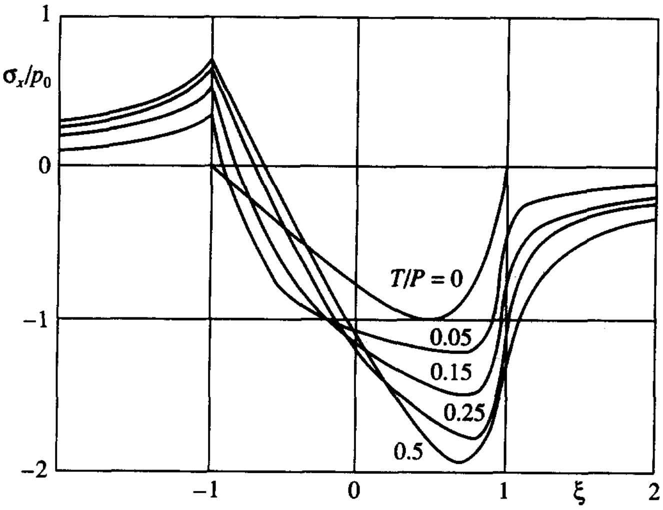

Fig. 9

properties of the surface layer and, in particular, by the model which is chosen to describe these properties. When the Kelvin model is used, this dependence is quite non-monotonic and the maximum value of the friction coefficient is reached at velocities of motion for which the time taken for the indentor to pass through the elementary contact spot is commensurate with the retardation time of the material of the surface layer [33, 34]. The results of the investigations in [33, 43] also indicate the substantial effect of the properties of the surface layer on the maximum shear and tensile stress distributions within the interacting bodies.

Hence, theoretical research on the contact interaction of bodies under conditions of rolling friction, at the source of which are A. Yu. Ishlinskii's papers, has progressed along a path of increasing complexity of the models of contacting bodies and the contact conditions. The results obtained in this field enable one to study the stressed state interacting bodies, which is important for the development of the theory of wear and contact-fatigue fracture of the surface layers of materials during rolling.

REFERENCES

1. ISHLINSKII, A., Yu., *Rolling Friction*. Candidate dissertation. Inst. Mat. Mekh. Mosk. Gos. Univ., Moscow, 1938.
2. ISHLINSKII, A. Yu., Rolling friction. *Prikl Mat. Mekh*., 1938, **2**, 2, 245–260.
3. ISHLINSKII, A. Yu., Theory of resistance to rolling (rolling friction) and related phenomena, In *Friction and Wear in Machines*. Izd. Akad. Nauk SSSR, Moscow and Leningrad, 1940, Vol. 2, pp. 255–264.
4. ISHLINSKII, A. Yu., Sliding in the contact region accompanying rolling friction. Izv. *Akad. Nauk SSSR. Otd. Tekh. Nauk*, 1956, 6, 3–15.
5. ISHLINSKII, A. Yu., *Mechanics: Ideas, Problems, Applications*. Nauka, Moscow, 1985.
6. ISHLINSKII, A. Yu., *Applied Problems of Mechanics*, Vol. 1, *Mechanics of Viscoplastic and Incompletely Elastic Bodies*; Vol. 2. *Mechanics of Elastic and Absolutely Rigid Bodies*. Nauka, Moscow, 1986.
7. COULOMB, Ch. A., Theorie des machines simples. *Mem. Math. et de Phys. de l'Acad. Roy. Sci.*, 1875, **10**, 161–331.
8. REYNOLDS, O., On rolling friction. *Phil. Trans. Roy. Soc. London A.*, 1877, **166**, 155–174.
9. PETROV, N. P., *Resistance of a Train on a Railway Track*. Tip. V. Demakova, St Petersburg, 1889.
10. TABOR, D., The mechanism of rolling friction. *Phil. Mag. Ser. 7*, 1952, **43**, 345, 1055–1059.
11. TOMLINSON, G. A., A molecular theory of friction. *Phil. Mag. Ser. 7*, 1928. **7**, 46, 905–939.
12. CARTER, F. W., On the action of a locomotive driving wheel. *Proc. Roy. Soc. London. Ser. A.*, 1926, **112**. 760, 151–157.
13. PORITSKY, H. V., Stresses and deflections of cylindrical bodies in contact with application to contact of gears and of locomotive wheel. *J. Appl. Mech.*, 1950, **17**, 2, 191–201.
14. CAIN, B. S., Discussion of the paper of Poritsky H. V. *J. Appl. Mech.*, 1950, **17**, 4, 465.
15. GLAGOLEV, N. I., Rolling resistance of cylindrical bodies. *Prikl. Mat. Mekh*., 1945, **9**, 4, 318–333.
16. JOHNSON, K. L., Tangential tractions and microslip in rolling contact. *Rolling Contact Phenomena* (Edited by Bidwell), Elsevier, New York; 1963, 25–28.
17. FROMM, H., Berechnung des Schlupfes beim Rollen deformierbarer Scheiben. *ZAMM*, 1927, **7**, 1, 27–58.
18. JOHNSON, K. L., *Contact Mechanics*. Cambridge, Univ. Press, New York; 1985.
19. KALKER, J. J., *Three-dimensional Elastic Bodies in Rolling Contact*. Dordrecht, Kluwer, 1990.
20. MOSSAKOVSKII, V. I. and MISHCHISHIN, I. I., Rolling of elastic bodies. *Prikl Mat. Mekh.*, 1867, **31**, 5, 870–876.
21. KALKER, J. J., A minimum principle of the law of dry friction, with application to elastic cylinders in rolling contact. Parts I–II. *Trans. ASME. Ser. E. J. Appl. Mech.*, 1971, **38**, 4, 875–887.

22. SPECTOR, A. A., Variational method for investigating contact problems with slip and stick, *Dokl. Akad. Nauk SSSR*, 1977, **236**, 1, 39–42.
23. GOL'DSHTEIN, R. V., ZAZOVSKII, A. F., SPEKTOR, A. A. and FEDORENKO, R. P., Solution by variational methods of rolling contact problems with slip and stick. *Uspekhi Mekhaniki*, 1982, **5**, 3/4, 61–102.
24. KALKER, J. J., A strip theory for rolling with slip and spin. *Proc. Kon. Ned. Akad. van Wetenschappen*., 1967, **70**, 1, 10–24.
25. DEPUIT, A. J. E. J., Sur le tirage des voitures et sur le frottement de roulement. *Ann. des Ponts et Chaussees,* 1842, **3**, 261–335.
26. MAY, W. D., MORRIS, E. L. and ATACK, D., Rolling friction of a hard cylinder over a viscoelastic material. *J. Appl. Phys*., 1959, **30**, 11, 1713–1724.
27. FLOM, D. G. and BUECHE, A. M., Theory of rolling friction for spheres. *J. Appl. Phys*., 1959, **30**, 11, 1725–1730.
28. HUNTER, S. C., The rolling contact of a rigid cylinder with a viscoelastic half space. *Trans. ASME. Ser. E. J. Appl. Mech*., 1961, **28**, 4, 611–617,
29. MORLAND, L. W., A plane problem of rolling contact in linear viscoelasticity theory. *Trans. ASME. Ser. E. J. Appl. Mech*. 1962, **29**, 2, 345–352.
30. MORLAND, L. W., Rolling contact between dissimilar viscoelastic cylinders. *Quart. Appl. Math.,* 1967, **25**, 4, 363–376.
31. IVANOVA, R. Ya., Rolling of a viscoelastic cylinder on a base of the same material. *Zh. Prikl. Mekh. Tekh. Fiz*., 1964, 3, 179–184.
32. GORYACHEVA, I. G., The contact problem of the rolling of a viscoelastic cylinder on a base of the same material. *Prikl. Mat. Mekh*., 1973, **37**, 5, 925–933.
33. GORYACHEVA, I. G., *Mechanics of Frictional Interaction*. Nauka, Moscow, 2001.
34. GALIN, L. A., *Contact Problems in the Theory of Elasticity and Viscoelasticity.* Nauka, Moscow, 1980.
35. HALLING, J. and AL-QISHTAINI, M. A., An experimental study of some of the factors affecting the contact conditions and the slip between a rolling ball and its track. *Proc. Inst. Mech. Engrs*., 1967–1968, **182**, 37, Part 1, 757–768.
36. LING, F. F. and LAI, W. M., Surface mechanics of layered media. *Solid Contact and Lubrication* (Edited by H. S. Cheng and L. M. Keer). ASME, New York, 1980, 27–30.
37. KALKER, J. J., Viscoelastic multilayered cylinders rolling with dry friction. *Trans. ASME. J. Appl. Mech*. 1991, **58**, 3, 666–679.
38. BRAAT, G. F. M. and KALKER, J. J., Theoretical and experimental analysis of the rolling contact between two cylinders coated with multilayered viscoelastic rubber. *Contact Mechanics, Computational Techniques* (Edited by M. H. Aliabadi and C. A. Brebbia). Contact Mechanics, 1993, 119–126.
39. SADEGI, F. and GORYACHEVA, I. G., Contact characteristics of rolling/sliding cylinder and a viscoelastic layer bonded to an elastic substrate. *Wear*, 1995, **184**, 2, 125–132.
40. GORYACHEVA, I. G., CORYACHEV, A. P. and SADEGI, F., The contacting of elastic bodies with thin viscoelastic coatings under conditions of rolling friction or slip. *Prikl. Mat. Mekh*., 1995, **59**, 4, 634–641.
41. GORYACHEVA, I. G., *Contact Mechanics in Tribology*. Kluwer, Dordrecht, 1998.
42. SPEKTOR, A. A., Slip and stick zones in the contact region of a rolling elastic cylinder and a base of the same material. *Izv. Akad. Nauk ArmSSr, Mekhanika*, 1975, **28**, 6, 60–65.
43. GORYACHEVA, I. G., ZAKHAROV, S. M. and TORSKAYA, Ye. V., The effect of relative slip and properties of the surface layer on the stressed state of elastic bodies in the case of rolling friction, *Treniye i Iznos*, 2003, **24**, 1, 5–15.

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